Geometric singular perturbation analysis of a Autocatalator model

Peter Szmolyan

Vienna University of Technology

joint work with: Ilona Kosiuk

Overview

- 2-d Autocatalator, slow-fast structure
- relaxation oscillation, asymptotics, numerics
- a rescaling
- two scaling Regimes
- blow-up analysis

Autocatalator Model

$$\begin{aligned} \dot{a} &= \mu - a - ab^2 \\ \varepsilon \dot{b} &= -b + a + ab^2 \end{aligned} \tag{1}$$

a slow, b fast, $0 < \varepsilon \ll 1$, parameter $\mu > 0$

fast time scale $\tau := t/\varepsilon$

$$\begin{aligned} a' &= \varepsilon(\mu - a - ab^2) \\ b' &= -b + a + ab^2 \end{aligned} \tag{2}$$

Slow-fast subsystems

The limiting systems for $\varepsilon = 0$:

- **the reduced problem**

$$\begin{aligned} \dot{a} &= \mu - a - ab^2 \\ 0 &= -b + a + ab^2 \end{aligned} \tag{3}$$

- **the layer problem**

$$\begin{aligned} a' &= 0 \\ b' &= -b + a + ab^2 \end{aligned} \tag{4}$$

Dynamics of the layer problem

critical manifold

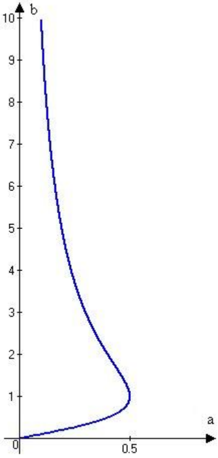
$$\mathcal{S} = \{a - b + ab^2 = 0\}$$

equilibria of

layer problem

$\mathcal{S}$ graph

$$a = \frac{b}{b^2+1}, \; b \geq 0$$



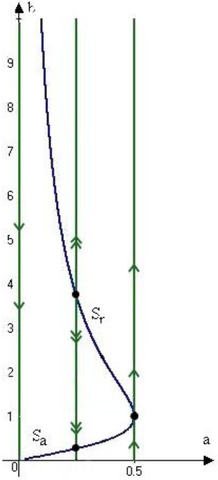
Fast dynamics

$\mathcal{S}$ has

attracting branch $\mathcal{S}_a$, $b < 1$

repelling branch $\mathcal{S}_r$, $b > 1$

non-hyperbolic fold point $p_f = (\frac{1}{2}, 1)$



Dynamics of the reduced problem

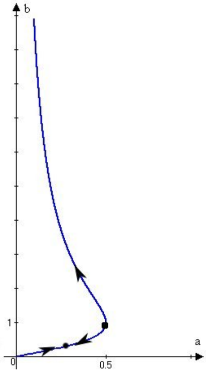
Differentiate $a = \frac{b}{1+b^2}$ with respect t

$$\dot{a} = \frac{1-b^2}{(1+b^2)^2}\dot{b} = \mu - b$$

- singular at $b = 1$, unless $\mu = 1$ (canard!)
- equilibrium $b = \mu$
- $\dot{a} > 0,\ b < \mu, \quad \dot{a} < 0,\ b > \mu$

Slow dynamics

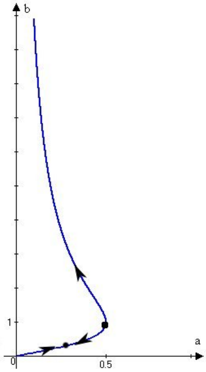
Three different cases:



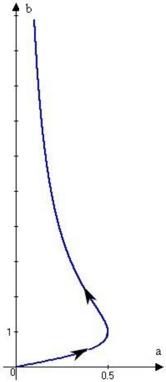
$\mu < 1$, excitable

Slow dynamics

Three different cases:



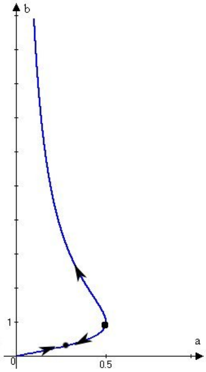
$\mu < 1$, excitable



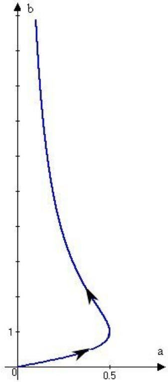
$\mu = 1$, canard

Slow dynamics

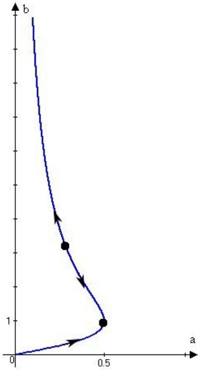
Three different cases:



$\mu < 1$, excitable



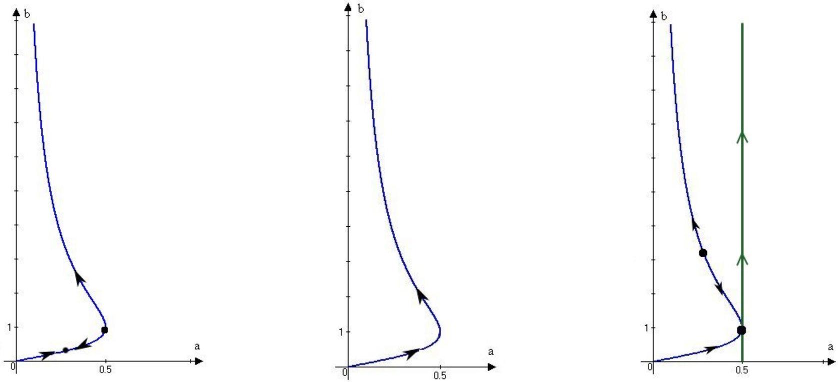
$\mu = 1$, canard



$\mu > 1$, jump point

Slow dynamics

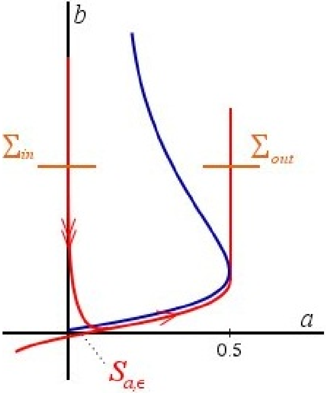
Three different cases:



$\mu < 1$, excitable $\mu = 1$, canard $\mu > 1$, jump point

For $\mu > 1$ we have a jump point at fold

good control for $\varepsilon \ll 1$, attraction onto $S_{a,\varepsilon}$
followed by jump, map: $\pi : \Sigma_{in} \to \Sigma_{out}$



Should we expect relaxation oscillation?

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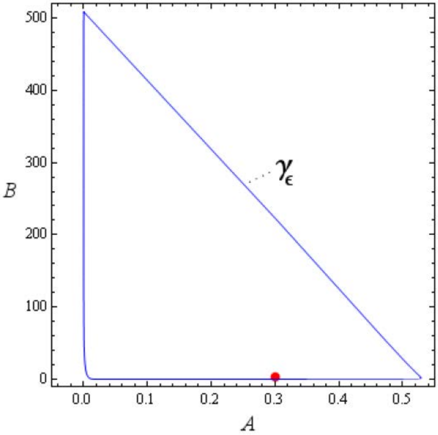
Let's ask the computer!

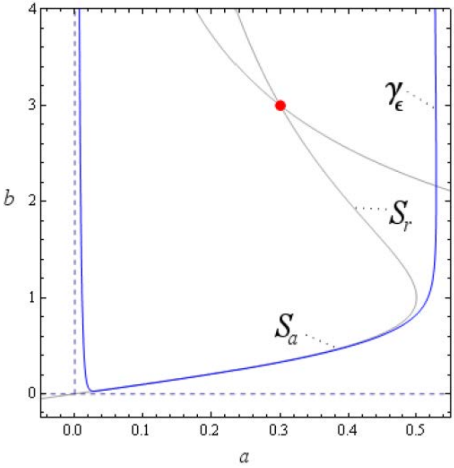
A rather "big" surprise



Computers also scale things!

rescale by zooming in!

This is closer to what we just proved



What did go wrong?

Let's check again what we did

$$
\begin{aligned}
a' &= \varepsilon(\mu - a - ab^2) \\
b' &= -b + a + ab^2
\end{aligned}
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a slow, only valid for $a,\ ab^2$ bounded!!!

Let's check again what we did

$$\begin{aligned} a' &= \varepsilon(\mu - a - ab^2) \\ b' &= -b + a + ab^2 \end{aligned}$$

a slow, only valid for $a,\ ab^2$ bounded!!!

a bounded, but b gets very large!

Rescaling for large b

GSPT valid for $b = O(1)$

$$
\begin{aligned}
a' &= \varepsilon(\mu - a - ab^2) \\
b' &= -b + a + ab^2
\end{aligned}
$$

- $b = O(1/\varepsilon) \Rightarrow$ new scales, different asymptotic analysis
- critical manifold not compact $\Rightarrow$ loss of normal hyperbolicity

Rescaling gives a super-fast system

New variables

$$a = A, \qquad b = \frac{B}{\varepsilon}, \qquad T = t/\varepsilon^2$$

Rescaled system

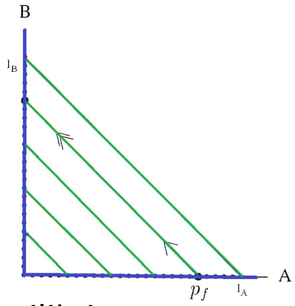
$$\begin{aligned} A' &= \mu\varepsilon^2 - A\varepsilon^2 - AB^2 \\ B' &= -B\varepsilon + A\varepsilon^2 + AB^2 \end{aligned} \tag{5}$$

where $'$ denotes differentiation with respect to T.

The rescaled layer problem is simple but degenerate

$\varepsilon = 0$ in rescaled system

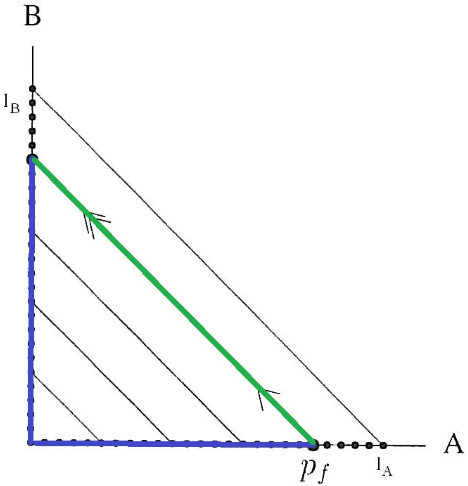
$$\begin{aligned} A' &= -AB^2 \\ B' &= AB^2 \end{aligned}$$



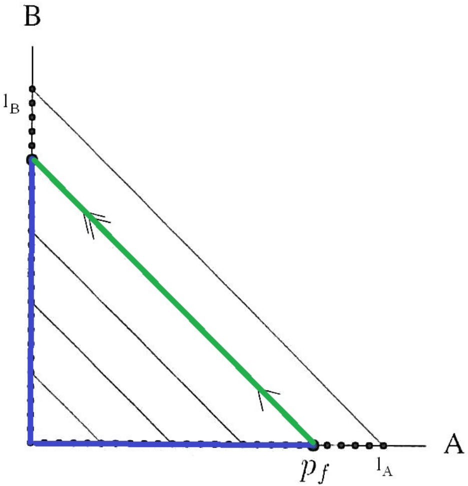
critical manifold: two lines of equilibria

- $A = 0$ normally hyperbolic, attracting
- $B = 0$ non-hyperbolic, weakly repelling

$\exists$ very degenerate singular periodic orbit γ_0



Critical manifold $A = 0$ is normally hyperbolic



Critical maifold $A = 0$ perturbs to slow manifold

$M_0 = \{(0, B) : B \in [\beta_0, \beta_1], \beta_0 > 0\}$ normally hyperbolic

Theorem: $\exists$ attracting slow manifold M_ε, given as

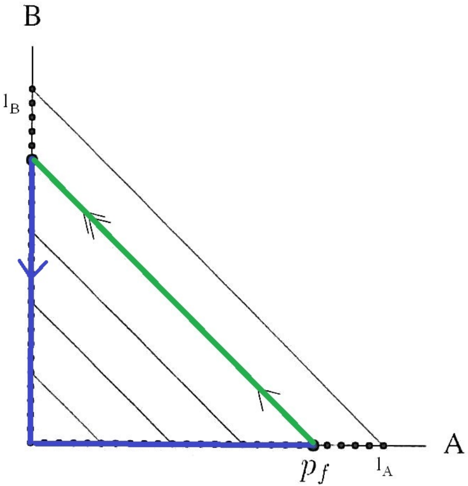
$$A = h(B, \varepsilon), \quad B \in [\beta_0, \beta_1]$$

with

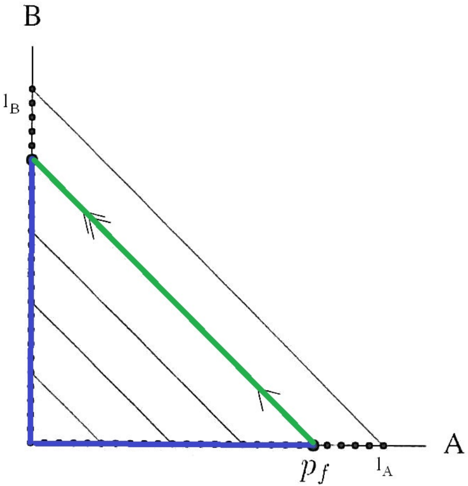
$$h(B, \varepsilon) = \varepsilon^2 \frac{\mu}{B^2} + O(\varepsilon^3) \qquad \text{singular as } B \to 0$$

slow flow on M_ε : $\quad \dfrac{dB}{d\tau} = -B + O(\varepsilon), \quad \tau = \varepsilon T = t/\varepsilon$

Return mechanism: reduced flow $B' = -B$

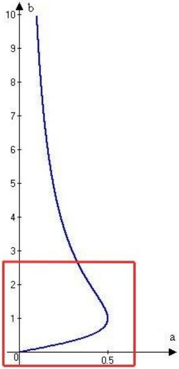


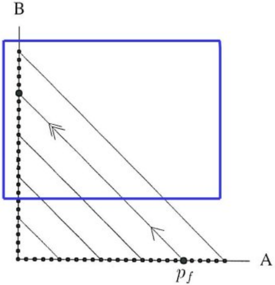
How did the folded critical manifold dissapear?



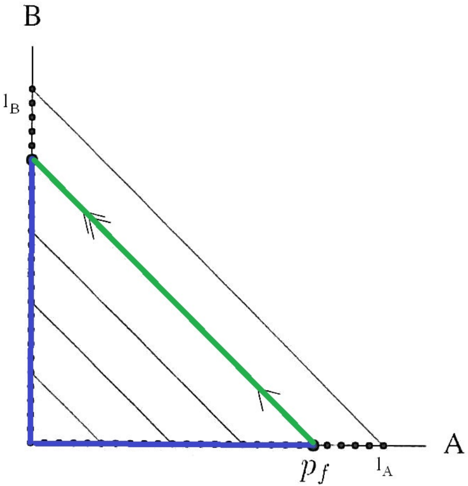
There exist two Regimes with good asymptotic analysis

Regime 1: $b = O(1)$ Regime 2: $B = O(1),\, b = 1/\varepsilon$



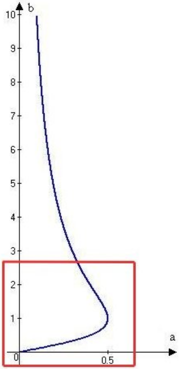


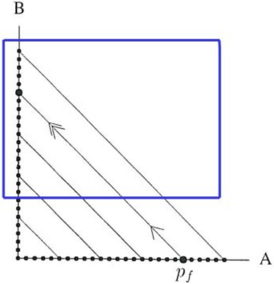
The fold is hidden in the nonhyperbolic line $B = 0$!!!



Can this be combined to prove existence of relaxation oscillation?

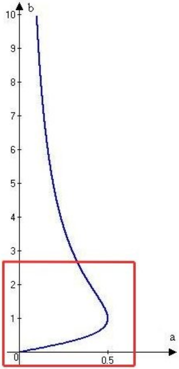
Regime 1: $b = O(1)$ Regime 2: $B = O(1)$, $b = 1/\varepsilon$

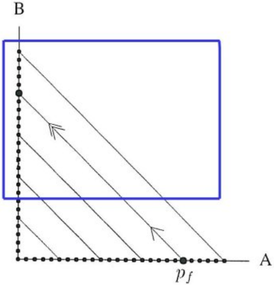




Overlap, matching, proof: ???????

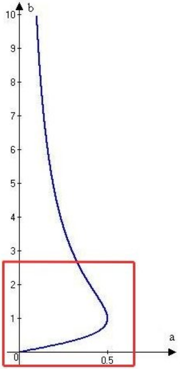
Regime 1: $b = O(1)$ Regime 2: $B = O(1)$, $b = 1/\varepsilon$

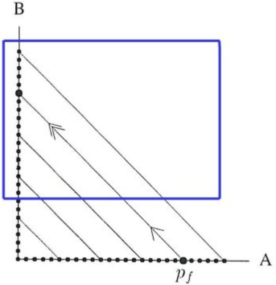




Overlap, matching, proof: blow-up

Regime 1: $b = O(1)$ Regime 2: $B = O(1)$, $b = 1/\varepsilon$





Main theorem

Theorem

For $\mu > 1$ and all ε sufficiently small there exists a unique periodic orbit γ_ε of system (5) and hence of the equivalent system (1) which tends to the singular cycle γ_0 for $\varepsilon \to 0$.

Proof: Blow-up analysis based on the extended system

$$\begin{aligned} A' &= \mu\varepsilon^2 - A\varepsilon^2 - AB^2 \\ B' &= -\varepsilon B + A\varepsilon^2 + AB^2 \\ \varepsilon' &= 0 \end{aligned} \tag{6}$$

- degenerate line l_A of equilibria: $B = 0$, $\varepsilon = 0$
- linearization at $(A, 0, 0)$: triple eigenvalue $\lambda = 0$

The degenerate line is blown-up to a cylinder

blow up transformation

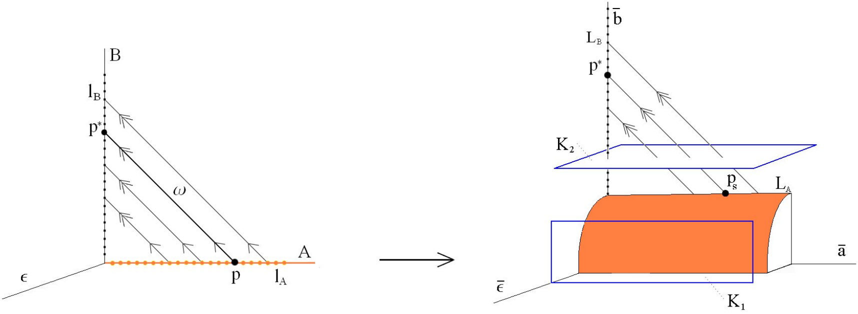
$$\begin{aligned} A &= \bar{a} \\ B &= r\bar{b} \\ \varepsilon &= r\bar{\varepsilon} \end{aligned} \tag{7}$$

with

$$\bar{a} \in \mathbb{R}, \quad \bar{b}^2 + \bar{\varepsilon}^2 = 1, \quad r \in \mathbb{R}$$

line l_A blown up to cylinder $\mathbb{R} \times S^1$, i.e. $r = 0$

Blow-up transformation and charts K_1 and K_2



Blow-up transformation and dynamics is described in charts

- chart K_1: $\bar{\varepsilon} = 1$, scaling chart

$$A = a_1, \quad B = r_1 b_1, \quad \varepsilon = r_1$$

- chart K_2: $\bar{b} = 1$, "compactification"

$$A = a_2, \quad B = r_2, \quad \varepsilon = r_2 \varepsilon_2$$

Chart K_1 covers Regime 1, i.e. the (a, b, ε) system

equations in chart K_1:

$$\begin{aligned} a_1' &= r_1(\mu - a_1 - a_1 b_1^2) \\ b_1' &= -b_1 + a_1 + a_1 b_1^2 \\ r_1' &= 0 \end{aligned}$$

This is the original system with

$$a = a_1, \quad b = b_1, \quad \varepsilon = r_1,$$

transforming to chart K_1 $\Leftrightarrow$ undoing rescaling

Chart K_2 covers Regime 2 and overlaps with Regime 1

equations in chart K_2

$$
\begin{aligned}
a' &= -r(a + \varepsilon^2 a - \varepsilon^2 \mu) \\
r' &= r(a + \varepsilon^2 a - \varepsilon) \\
\varepsilon' &= -\varepsilon(a + \varepsilon^2 a - \varepsilon)
\end{aligned} \tag{8}
$$

here $'$ denotes differentiation with respect to a rescaled time variable t_2.

In chart K_2 we meet old friends!

invariant subspaces

drop subscript "2"

- $\varepsilon = 0$

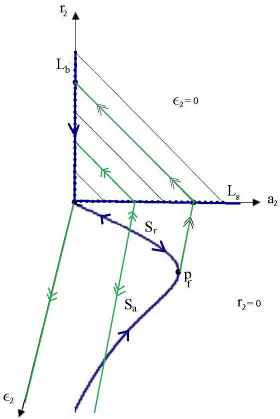
$$\begin{aligned} a' &= -ar \\ r' &= ar \end{aligned}$$

critical manifolds

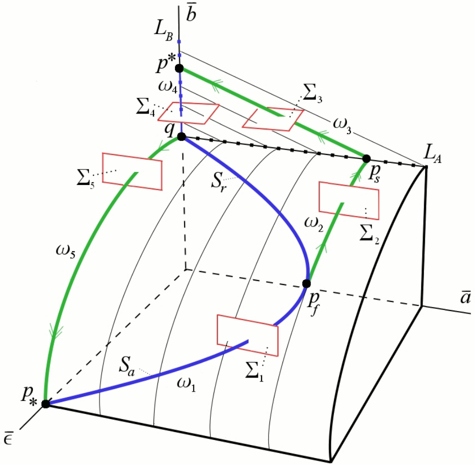
L_s, L_b, S

- $r = 0$

$$\begin{aligned} a' &= 0 \\ \varepsilon' &= (\varepsilon - a - \varepsilon^2 a)\varepsilon \end{aligned}$$



In the blown-up space there exists a desingularized singular cycle $\Gamma_0 = \omega_1 \cup \omega_2 \cup \omega_3 \cup \omega_4 \cup \omega_5$



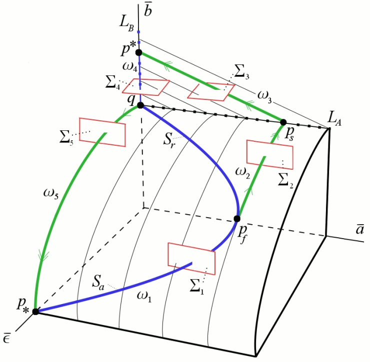
Existence of the limit cycle

Theorem

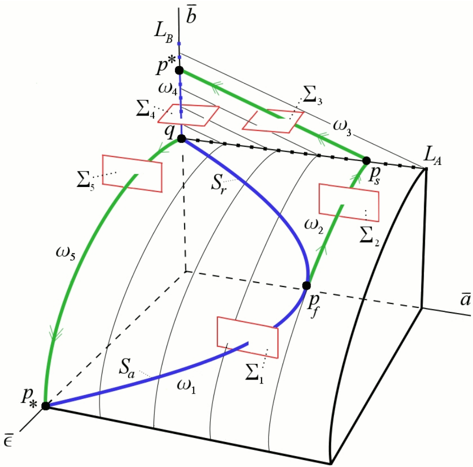
For sufficiently small ε the blown up vector field $\bar{X}$ has a family of periodic orbits $\bar{\Gamma}_\varepsilon$ which for $\varepsilon = r\bar{\varepsilon} = 0$ tend to the singular cycle $\Gamma_0 = \omega_1 \cup \omega_2 \cup \omega_3 \cup \omega_4 \cup \omega_5$.

Proof is based on Poincare map $\Pi : \Sigma_1 \to \Sigma_1$

$\Pi_1 : \Sigma_1 \to \Sigma_2$	–	passage of the fold point p_f
$\Pi_2 : \Sigma_2 \to \Sigma_3$	–	passage of the hyperbolic line L_s
$\Pi_3 : \Sigma_3 \to \Sigma_4$	–	contraction and slow drift toward the vertical slow manifold
$\Pi_4 : \Sigma_4 \to \Sigma_5$	–	passage of the nilpotent point q
$\Pi_5 : \Sigma_5 \to \Sigma_1$	–	transition towards the attracting slow manifold.



Proof uses (known) blow-up of fold-point p_f and needs new (simple) blow-up of point q



Why did it work?

- desingularization by blow up
- hidden details are visible
- blow-up gives scaling and overlap
- perturb from a well behaved limiting object
- gain of hyperbolicity
- GSPT becomes applicable